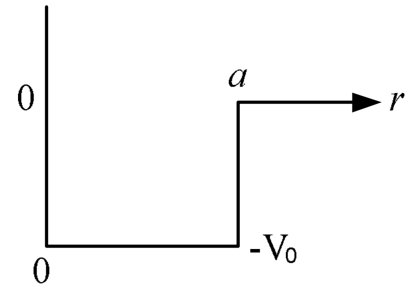
The scattering problem for quantum mechanics I



Problem:
Find the cross-section for the scattering of low velocity particles by a potential well (The de Broglie wavelength is large enough compared with the structure of the potential.)

Solution:
It is required to obtain the phase shift to have the cross section. Start with the asymptotic form of the radial functions satisfying the equation:

$$r > a: \quad \chi_l'' + \left[k^2 - \frac{l(l+1)}{r^2}\right]\chi_l = 0 \quad \text{with } k^2 = \frac{2\mu E}{\hbar^2}$$

and

$$r < a: \quad \chi_l'' + \left[k'^2 - \frac{l(l+1)}{r^2}\right]\chi_l = 0 \quad \text{with } k'^2 = \frac{2\mu(E - V_0)}{\hbar^2}$$

This system requires the boundary condition, $\chi_l(0) = 0$, assuming that the center is also bounded.
When the de Broglie wavelength is considerably larger than the dimensions of the well, the major contribution to the scattering arises from the S-wave. The asymptotic wave function, χ_0, which satisfies the boundary condition, will be expressed as follows:

$$\chi_0 = A \sin k'r \quad (r < a)$$

$$\chi_0 = \sin(kr + \delta_0) \quad (r > a)$$

The phase, δ_0, and the coefficient, *A*, follow from the condition that both the wave function and its derivative are continuous at *r*=*a*. Then, we have

$$\delta_0 = \arctan\left(\frac{k}{k'}\tan k'a\right) - ka$$

The partial cross-section without angular momentum, *l*=0, is expressed as

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0$$

Therefore,

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\left[\arctan\left(\frac{k}{k'}\tan k'a\right) - ka\right]$$

For small velocities of the incident particles, when *k* is close to zero, the phase, δ_0, will be linearly proportional to *k*:

$$\delta_0 \approx k\left(\frac{\tan k_0 a}{k_0 a} - 1\right) \quad \text{with } k_0^2 = \frac{2\mu V_0}{\hbar^2}$$

Due to the factor, $\frac{1}{k^2}$, the cross-section will be

$$\sigma \approx 4\pi a^2 \left(\frac{\tan k_0 a}{k_0 a} - 1 \right)^2 \qquad \text{(For small } k\text{)}$$

We consider the cross-section as a function of the depth of the potential to determine k_0. If the potential well is shallow; namely, $k_0 a << 1$, we have

$$\sigma = 4\pi a^2 \frac{k_0^4 a^4}{9} = \frac{16\pi}{9} \frac{a^6 V_0^2 \mu^2}{\hbar^4}$$

From the perturbation theory, we obtain

$$f(\vartheta) = -\frac{1}{4\pi} \frac{2\mu}{\hbar^2} \int V(r) d\tau = \frac{2\mu}{\hbar^2} V_0 \frac{a^3}{3}$$

Therefore, the cross-section is

$$\sigma = 4\pi |f(\vartheta)|^2 = \frac{16\pi}{9} \frac{a^6 V_0^2 \mu^2}{\hbar^4}$$

The cross-section increases as V_0 increases and it diverges when $k_0 a = \pi/2$ that is the first level of the potential well. The deeper the potential becomes, the smaller the cross-section will be. Then, it becomes zero when $\tan k_0 a = k_0 a$. When V_0 is much larger, the cross-section oscillates between 0 and ∞. It becomes infinity when *a* goes to a new level.